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Indicated coloring of graphs

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ABSTRACT

We study a graph coloring game in which two players collectively color the vertices of a graph in the following way. In each round the first player (Ann) selects a vertex, and then the second player (Ben) colors it properly, using a fixed set of colors. The goal of Ann is to achieve a proper coloring of the whole graph, while Ben is trying to prevent realization of this project. The smallest number of colors necessary for Ann to win the game on a graph G (regardless of Ben's strategy) is called the *indicated chromatic number* of G, and denoted by $\chi_i(G)$. We approach the question how much $\chi_i(G)$ differs from the usual chromatic number $\chi(G)$. In particular, whether there is a function f such that $\chi_i(G) \leqslant f(\chi(G))$ for every graph G. We prove that f cannot be linear with leading coefficient less than $4/3$. On the other hand, we show that the indicated chromatic number of random graphs is bounded roughly by $4\chi(G)$. We also exhibit several classes of graphs for which $\chi_i(G) = \chi(G)$ and show that this equality for any class of perfect graphs implies Clique-Pair Conjecture for this class of graphs.

1. Introduction

Suppose that two players, Ann and Ben, are jointly coloring a graph G using a fixed set of colors C. In each round Ann picks an uncolored vertex and colors it, and then Ben is doing the same. They both agree on respecting the rule of proper coloring: none of them is allowed to create a monochromatic edge. The game stops when either the whole graph is successfully colored, or all colors from C occur on vertices adjacent to some uncolored vertex. In the former case Ann is the winner, in the latter case—Ben. The minimum size of color set C guaranteeing a win for Ann is called the *game chromatic number* of a graph G, and is denoted by $\chi_g(G)$.

The idea of game coloring in the above form was introduced independently by Bodlaender [4] and Gardner [9]. It was originally motivated by the four color problem and computational complexity issues. Since then the topic developed into several directions leading to deep results, sophisticated methods, and challenging open problems (see a recent survey [3]). There are also some unexpected connections to other areas, such as, for instance, the surprising application of game coloring to graph packing discovered by Kierstead and Kostochka [12].

In this paper we study a variant of the graph coloring game proposed by Grytczuk [10]. In this modification the roles of players are highly asymmetric: in one round Ann is only picking a vertex while Ben is choosing a color for this vertex. All other rules and goals of the players remain the same. So, Ben is not allowed to create a monochromatic edge, but tries to "block" some vertex by using all colors from C on its neighbors before Ann will pick it. The minimum number of colors needed for Ann to win this game on a graph G is denoted by $\chi_i(G)$, and is called the *indicated chromatic number* of G.

At first glance $\chi_i(G)$ behaves more tamely than $\chi_g(G)$. Indeed, it is not hard to see that for bipartite graphs we have $\chi_i(G) = 2$, while $\chi_g(G)$ takes arbitrarily large values in this class. Also $\chi_i(G) \leqslant \mathrm{col}(G)$ (the *coloring number* of G, defined precisely in Section 5), which is far from the truth for $\chi_g(G)$. The question we approach is whether the indicated chromatic

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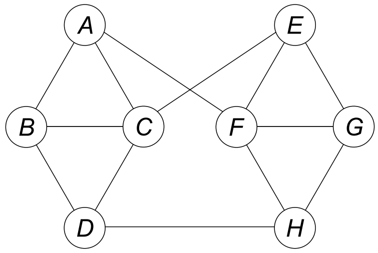

Fig. 1. Twisted diamond graph J for which $\chi_i(J) > \chi(J)$.

number is χ-bounded. More precisely, let $I(n)$ be the supremum of the values $\chi_i(G)$ for graphs satisfying $\chi(G) \leqslant n$, where $\chi(G)$ denotes the chromatic number of G. We believe that $I(n)$ is always finite. We show that the indicated chromatic number of the random graph $G_{n,p}$ is bounded by roughly $4\chi(G_{n,p})$. We also examine a strange Zhu-type question [14] (asked originally for the game chromatic number) whether enlarging the number of colors could be in favor of Ben. We show that there is a graph H such that $\chi_i(H) = 3$, but if there are four colors available, then it is much harder for Ann to win the game on H. In the final section we present some connection of the indicated chromatic number of perfect graphs with the Clique-Pair Conjecture stated by Fonlupt and Sebö [7].

2. Upper bound

We start with showing that the difference between $\chi_i(G)$ and $\chi(G)$ can be arbitrarily big. For this purpose we need a graph G for which $\chi_i(G)$ is strictly bigger than $\chi(G)$. As $\chi_i(G) = \chi(G)$ for every bipartite graph G (which is easy to see), we made a search among 3-chromatic graphs. One of the simplest examples is depicted in Fig. 1. We call this graph the *twisted diamond* and denote it by the letter J. This graph was found by Hałuszczak by computer search.

Lemma 1. *The twisted diamond graph satisfies $\chi_i(J) = 4$.*

Proof. First notice that J is uniquely 3-colorable. This is so because any proper coloring of the subgraph induced by the vertices $ABCD$ requires that A and D has the same color. Similarly E and H must have the same color. Vertices D and H are adjacent, so their colors should be different. Now, it is easy to see that there is only one way to put colors on the other vertices. We need to prove that there is no winning strategy for Ann when the game is played using three colors. By the unique colorability of J, Ann cannot leave any decision to Ben. Otherwise Ben could choose a wrong color which violates the unique coloring of J, and win the game in consequence. This type of Ann's moves will be called *forcing* moves. We will prove that such forcing play is impossible for Ann. Let J_1 denote the induced subgraph on vertices A, B, C, D and let J_2 be the induced subgraph on vertices E, F, G and H. Ann cannot start with two non-adjacent vertices. If Ann starts with two adjacent vertices from one of these subgraphs, she can color this subgraph using forcing moves, but then she cannot do any forcing move to the second of these subgraphs. If Ann starts with two adjacent vertices, but one in J_1 and the other in J_2, she also cannot do a forcing move, because there are no triangles containing an edge between J_1 and J_2. This completes the proof. □

Using this lemma we can prove the aforementioned lower bound on the function $I(n)$.

Theorem 2. *The function $I(n)$ satisfies $I(n) \geqslant \frac{4}{3}n$ for every n divisible by 3.*

Proof. If we take a graph J_k as k-copies of the twisted diamond graph J connected by all the possible edges, from the above lemma we get $\chi_i(J_k) = 4k$ for every $k \geqslant 1$. This completes the proof as $\chi(J_k) = 3k$. □

This theorem excludes the possibility of a linear upper bound for $I(n)$ with leading coefficient less than 4/3, but perhaps $I(n) \leqslant 2n$. We think that $I(n)$ is always finite, but we are not aware of a proof of that even for $n = 3$. On the other hand, no 3-chromatic graph is known with $\chi_i(G) \geqslant 5$. The hardness of determining the indicated chromatic number may be partially explained by the fact that this number is not monotonic with respect to taking subgraphs. Indeed, the twisted diamond J is a subgraph of the full tripartite graph $K_{3,3,3}$, which has $\chi_i(K_{3,3,3}) = 3$ which is less than $\chi_i(J) = 4$.

Even if we restrict to induced subgraphs, then a similar obstacle holds. Consider a twisted diamond J with a new vertex I connected to vertices A, B, D, E, G and H. To win with three colors Ann presents vertex I, which gets the first color, and then indicates vertex A, which gets a different color. In subsequent moves Ann can always indicate a vertex which is connected to two vertices having different colors and not connected with a vertex in the third color. It forces all Ben's choices and ends up with good coloring of the whole graph using 3 colors.

3. Random graphs

Let $G_{n,p}$ stand for the probability space of all labeled graphs on n vertices, where every edge appears independently with probability p (see [1,11]). A sequence of events X_n occurs *with high probability* if $\lim_{n\to\infty} \mathbb{P}(X_n) = 1$. We will show that the indicated chromatic number of $G_{n,p}$ is linearly bounded in terms of the chromatic number of $G_{n,p}$. We adopt the method used by Bohman et al. in [5] to prove a similar result for the game chromatic number $\chi_g(G_{n,p})$. Define $b = \frac{1}{1-p}$.

Theorem 3. *If $\varepsilon > 0$ is a constant, then with high probability*

$$\chi_i(G_{n,p}) \leqslant (2+\varepsilon)\frac{n}{\log_b np}.$$

Proof. Let the number of colors be $k = (2+\varepsilon)\frac{n}{\log_b np}$ and let $\mathcal{C} = (C_1, C_2, \dots, C_k)$ be a collection of pairwise disjoint sets of vertices. For a vertex v let

$$A(v,\mathcal{C}) = \{1 \leqslant i \leqslant k : v \text{ is not adjacent to any vertex of } C_i\}$$

and set

$$a(v,\mathcal{C}) = |A(v,\mathcal{C})|.$$

Note that $A(v,\mathcal{C})$ is the set of available colors for uncolored vertex v when the partial coloring is given by the sets in $\mathcal{C}$.

Let

$$\alpha = 2+\varepsilon, \qquad \beta = k(1-p)^{n/k} = \frac{\alpha n}{(np)^{1/\alpha}\log_b np}, \qquad \gamma = \frac{9n\ln n}{\beta},$$

and

$$B(\mathcal{C}) = \{v : a(v,\mathcal{C}) < \beta/2\}.$$

We will show that with high probability every partial coloring of the vertex set has the property that there are at most γ vertices with less than $\beta/2$ available colors.

Claim 4. *With high probability, for all partial collections $\mathcal{C}$*

$$|B(\mathcal{C})| \leqslant \gamma.$$

Proof. For some fixed $\mathcal{C}$ and every uncolored vertex v, the number of available colors is the sum of independent variables X_i, where $X_i = 1$ if v has no neighbors in C_i. Then $\mathbb{P}(X_i = 1) = (1-p)^{|C_i|}$, and since $(1-p)^x$ is a convex function we have

$$\begin{aligned}\mathbb{E}(a(v,\mathcal{C})) &= \sum_{i=1}^{k}(1-p)^{|C_i|} \geqslant k(1-p)^{(|C_1|+\cdots+|C_k|)/k}\\ &\geqslant k(1-p)^{n/k} = \beta.\end{aligned}$$

It follows from the Chernoff bound (see [1] or [11]) that

$$\mathbb{P}(a(v,\mathcal{C}) \leqslant \beta/2) \leqslant e^{-\beta/8}.$$

Thus,

$$\mathbb{P}\left(\exists\mathcal{C} \text{ with } |B(\mathcal{C})| > \gamma\right) \leqslant k^n \binom{n}{\gamma} e^{-\beta\gamma/8} = o(1). \quad \square$$

Set t_0 to be the last time for which Ann presents a vertex with at least $\beta/2$ available colors, i.e.

$$t_0 = \min\{t : a(v,\mathcal{C}_t) \geqslant \beta/2 \text{ for all } v\},$$

where $\mathcal{C}_t$ denotes the collection of color classes when t vertices remain uncolored. From the above lemma we get $t_0 \leqslant \gamma$. It means that at some point where the number of uncolored vertices is less than γ, every vertex still has at least $\beta/2$ available colors. In particular, if $\beta/2 > \gamma$, then Ann will win the game since no vertex will ever run out of colors. It can be easily calculated that condition $\beta/2 > \gamma$ holds for n big enough. $\square$

It is well known by the results of Bollobás [6] and Łuczak [13] that $\chi(G_{n,p}) = (1+o(1))\frac{n}{2\log_b np}$ holds with high probability. So, our result shows that the indicated chromatic number is (up to a multiplicative constant) of the same asymptotic order as the chromatic number.

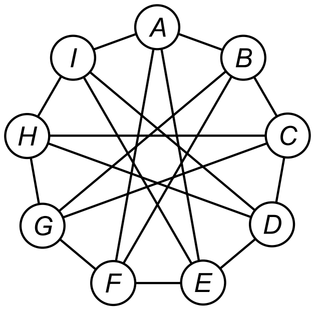

Fig. 2. Sunflower graph S.

4. The more colors, the easier to color?

In [14] Zhu asked the following question. Suppose that Ann has a winning strategy in the coloring game on a graph G with k colors. Is it true that she has a winning strategy on the same graph G using $k+1$ colors? The same Zhu-type question can be asked for the indicated coloring game. The question seems to have an obvious positive answer: the more colors, the easier to color a graph. But on the other side, if Ben has more colors at his disposal, then Ann has a weaker chance to locally force desired colors.

We prove first that this question has a positive answer for bipartite graphs. Then we shall show an example of a graph on which Ann wins easily with three colors, but has to be very careful when four colors are in the game.

Theorem 5. *For every bipartite graph G and for each $k \geqslant 2$, Ann has a winning strategy in the indicated coloring game on G.*

Proof. Let X and Y be the bipartition classes of G, and let the set of colors be $C = \{1, 2, \dots, k\}$. Ann starts with presenting some vertex $x \in X$, and suppose that Ben colors it by 1. Next Ann presents all neighbors of x in Y, and they get some colors from $C \setminus \{1\}$. In subsequent rounds Ann presents all neighbors of those vertices which have color 2. They can get color 1 or at least 3. Then she repeats this strategy of showing alternately neighbors of vertices colored by 1 and neighbors of vertices colored by 2. There cannot be any problem because no vertex in Y has color 1 and no vertex in X has color 2. This will stop when all neighbors of all vertices colored 1 or 2 are colored. At this point, vertices colored 1 or 2 cannot be a part of bad coloring (they cannot have a "blocked" neighbor, i.e., an uncolored vertex adjacent to all colors), because all of their neighbors are already colored. So, we can forget about them. Now Ann can start her strategy from the beginning by presenting some vertex in X and, if Ben colors it by 1 or 2, Ann should present neighbors of 1's and 2's. Similarly, new vertices colored 1 or 2 are in different parts of the partition (not necessarily the same parts as the old vertices colored 1 and 2). Thus, there cannot occur a bad partial coloring. □

Suppose now that we impose an additional restriction for Ann: she may pick a vertex only if it is adjacent to some colored vertex (except the first move which is arbitrary). Intuitively, it is not a restriction, because if Ann makes an "unconnected" move, she leaves Ben a full choice of colors. In many examples of graphs a winning strategy for Ann satisfies this connectivity restriction. Consider for instance the *sunflower* graph S depicted in Fig. 2. This graph was found by Hałuszczak by computer search. It is easy to check that $\chi_i(S) = 3$, and that a connected strategy works. In view of this the following proposition looks surprising.

Proposition 6. *If Ann uses any connected strategy on the sunflower graph S, then Ben wins the game on S with 4 colors.*

Proof. The strategy for Ben is: if selected vertex v has colored vertex u at distance 2 on the border cycle in S and also common neighbor w between these vertices is colored or has all neighbors except v already colored, then vertex v gets the color of vertex u. Otherwise (or when it is not possible) it gets a new color. It can be checked that if Ann makes only moves with the connectivity restriction, then Ben can play according to this strategy until there appears an uncolored vertex which has neighbors colored by all colors. There must appear such a vertex because each color can be used at most two times (if vertices in distance 2 on the border cycle has the same color, this color cannot be used anymore) and there are 4 colors and 9 vertices. □

Proposition 7. *There exists a winning strategy for Ann on the sunflower graph S with 4 colors.*

Proof. The sunflower S is a regular graph of degree 4. Hence, if some vertex has two neighbors sharing the same color, or we know that it will be indicated by Ann before some of his neighbors, then it will be able to be colored. Such a vertex will be called *safe*. If at some moment of the game all vertices are already colored or safe, then Ann wins the game.

In the first move Ann presents vertex A. Assume that Ben puts color 1. In the second move, Ann indicates vertex C. Ben can put color 1, or a new color, say color 2. In the first case, vertex B will be safe. So, Ann can present it at the very end of the game. Hence vertices F and G are safe. In the same way vertices E, H and vertices D, I are safe. In other words, if Ann

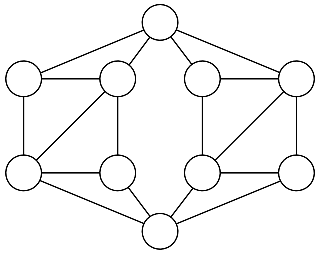

Fig. 3. An example of a perfect graph W with $\chi_i(W) > \chi(W)$.

indicates vertices in order D, I, E, H, F, G, B, the game will end in a good coloring. In the second case, Ann presents vertex B, which has to get a new color, say 3. Next, Ann indicates vertices F and G. Whatever Ben does, vertices C and F will have the same color 2, or vertices A and G will have color 1. These are symmetric cases, so we can assume the first one holds. Notice that vertex I is safe because none of its neighbors can have color 2. So, its neighbors are also safe. All of the vertices are colored or safe, hence the game will end in good coloring. □

This proves that in a game with the connectivity restriction the answer for this Zhu-type question is negative—the larger number of colors may gives Ben a winning strategy.

5. Perfect graphs

We conclude the paper with an intriguing question concerning perfect graphs. Let G_L be a graph with some linear order L on the set of vertices. For a vertex v, let $d_L^+(v)$ denote the number of neighbors of v that precede v in this order. One may think of the orientation of G obtained by drawing a sign $<$ on every edge uv whenever $u < v$ in the order L. Then $d_L^+(v)$ is just the outdegree of the vertex v in this orientation. Let $\Delta^+(G_L)$ denote the maximum outdegree in this orientation. The *coloring number* of a graph G is defined by $\mathrm{col}(G) = \min_L\{\Delta^+(G_L)\} + 1$, where the minimum is taken over all linear orders of $V(G)$.

It is obvious that $\chi_i(G) \leqslant \mathrm{col}(G)$ (just let Ann pick vertices in the order realizing $\mathrm{col}(G)$). This gives, for instance, that $\chi_i(G) \leqslant 6$ for every planar graph G. So, for every graph satisfying $\chi(G) = \mathrm{col}(G)$ we get $\chi(G) = \chi_i(G)$. This happens for chordal graphs and one is tempting to conjecture that, perhaps, the equality holds for other classes of perfect graphs.

It is surprising that this problem is connected to the well-known Clique-Pair Conjecture. To introduce this conjecture we need to make some definitions. We will call a graph *uniquely colorable* if there is only one partitioning to color classes in proper coloring. By a *clique-pair* of size ω we will understand a clique of size $\omega + 1$ without a single edge (it is also two cliques of size ω whose intersection is a clique of size $\omega - 1$).

Conjecture 8 (*Clique-Pair Conjecture CPC*)**.** *If G is a uniquely colorable perfect graph, and not a clique, then it contains a clique-pair of size $\omega(G)$.*

This conjecture was stated by Fonlupt and Sebö [7] in 1990 and is still open. It was verified for 3-chromatic graphs and for some easy classes of graphs. For more information about CPC see [2,8].

Proposition 9. *If G is a uniquely colorable perfect graph with $\chi_i(G) = \chi(G)$, and not a clique, then it contains a clique-pair of size $\omega(G)$.*

Proof. Since graph G is uniquely colorable, all moves of Ann must be forcing moves. Since $\chi_i(G) = \chi(G)$, there is an ordering in which all moves are forcing. It means that the first $\omega(G)$ vertices in this ordering form a clique and the next vertex forms a clique with some $\omega(G) - 1$ vertices among the previous ones. Hence, the first $\omega(G) + 1$ vertices forms a clique-pair.

This result means that if for some class of uniquely colorable perfect graphs $\chi_i(G) = \chi(G)$, then CPC is true for this class of graphs.

Question 10. *For which classes $\mathcal{G}$ of uniquely colorable perfect graphs, every graph $G \in \mathcal{G}$ satisfies $\chi(G) = \chi_i(G)$?*

Besides chordal graphs and bipartite graphs, the above equality holds for comparability graphs and their complements, as checked by Grytczuk [10]. Hence, CPC is true for this class of graphs. Unfortunately this equality does not hold for every uniquely colorable perfect graph. A counterexample for this fact was found by Grytczuk [10] and is showed on Fig. 3.

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